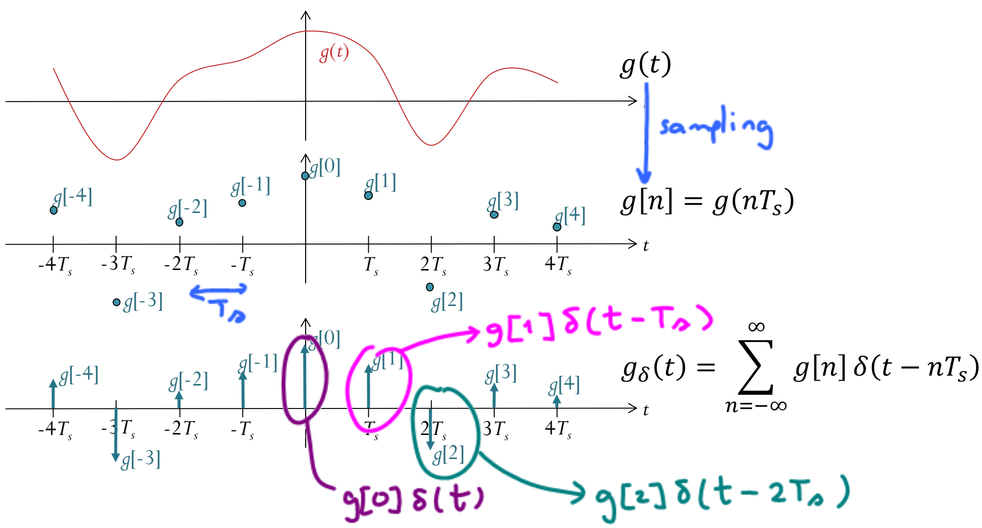
6.21. For the remaining analysis in this chapter, we will use $g(t)$ to denote the signal under consideration. You may replace $g(t)$ below by $m(t)$ if you want to think of it as an analog message to be transmitted by a communication system. We use $g(t)$ here because the results provided here work in broader setting as well.

6.2 Ideal Sampling

Definition 6.22. In **ideal sampling**, the (ideal instantaneous) sampled signal is represented by a train of impulses whose areas equal the instantaneous sampled values of the signal

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s).$$



6.23. The Fourier transform $G_\delta(f)$ of $g_\delta(t)$ can be found by first rewriting $g_\delta(t)$ as

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} g(t)\delta(t - nT_s)$$
$$= g(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s).$$

Multiplication in the time domain corresponds to convolution in the frequency domain. Therefore,

$$G_\delta(f) = \mathcal{F}\{g_\delta(t)\} = G(f) * \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right\}.$$

For the last term, the Fourier transform can be found by applying what we found in Example 4.47[25]:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s).$$

This gives

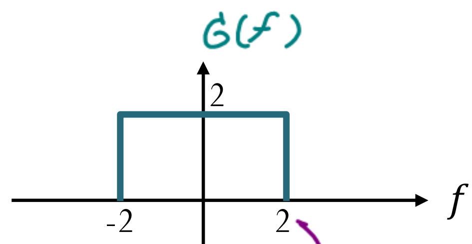
$$G_\delta(f) = G(f) * f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s) = f_s \sum_{k=-\infty}^{\infty} G(f) * \delta(f - kf_s).$$

Hence, we conclude that

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_\delta(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s). \quad (83)$$

In words, $G_\delta(f)$ is simply a sum of the scaled and shifted replicas of $G(f)$.

Example 6.24. Consider a continuous-time signal $g(t)$ whose Fourier transform is plotted below.



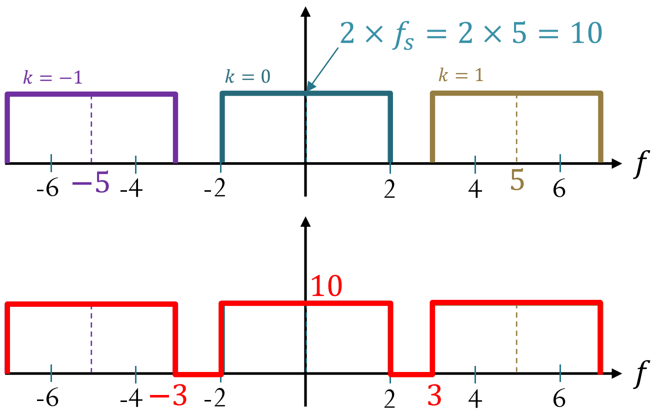
(a) Find the Nyquist sampling rate for this signal.

[25] We also considered an easy-to-remember pair and discuss how to extend it to the general case in 4.48.

$G_\delta(f)$

(b) Plot the Fourier transform of $g_\delta(t)$ from $f = -6$ to $f = 6$

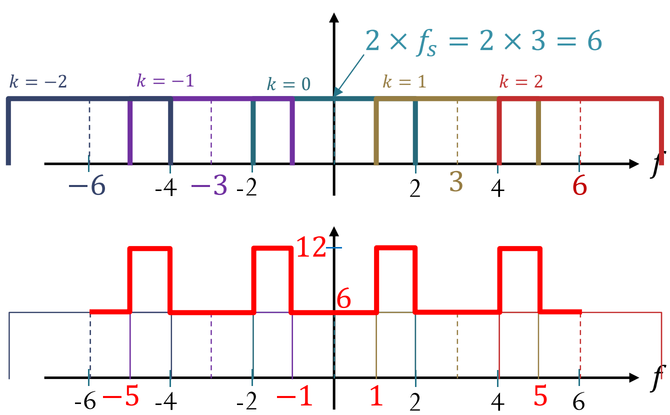
(i) when the sampling interval is $T_s = \frac{1}{5}$ $\Rightarrow f_s = \frac{1}{T_s} = 5$ [sa/s]



$$G_\delta(f) = \sum_{k=-\infty}^{\infty} f_s G(f - k f_s)$$

Only $f_s G(f - k f_s)$ for $k = -1, 0, 1$ are shown here. The contribution from other k values are outside of this specified freq. range.

(ii) when the sampling interval is $T_s = \frac{1}{3}$ $\Rightarrow f_s = \frac{1}{T_s} = 3$ [sa/s]



$$G_\delta(f) = \sum_{k=-\infty}^{\infty} f_s G(f - k f_s)$$

Only $f_s G(f - k f_s)$ for $k = 0, \pm 1, \pm 2$ are shown here. The contribution from other k values are outside of this specified freq. range.

6.25. As usual, we will assume that the signal $g(t)$ is band-limited to B Hz (($G(f) = 0$ for $|f| > B$)).

(a) When $B < f_s/2$ as shown in Figure 50, the replicas do not overlap and hence we do not need to spend extra effort to find their sum.

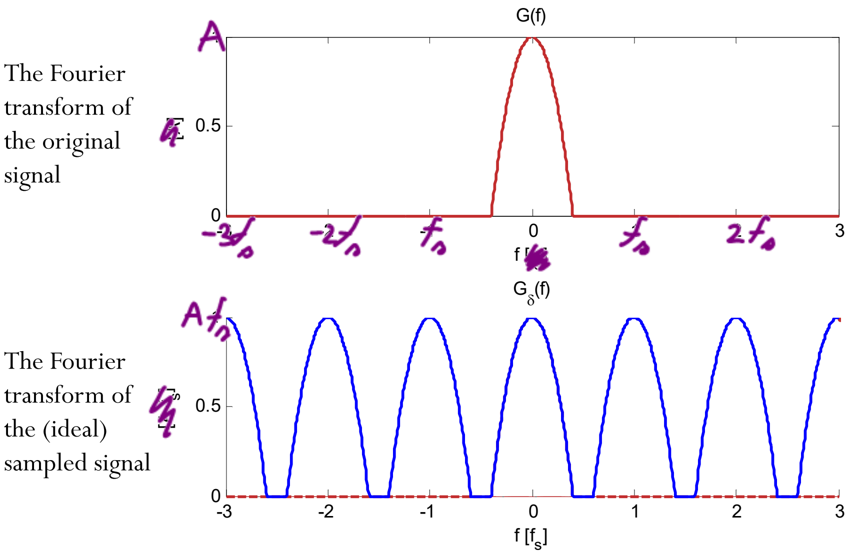


Figure 50: The Fourier transform $G_\delta(f)$ of $g_\delta(t)$ when $B < f_s/2$

(b) When $B > f_s/2$ as shown in Figure 51, overlapping happens in the frequency domain. This spectral overlapping of the signal is (also) commonly referred to as "aliasing" mentioned in 6.7. To find $G_\delta(f)$, dont forget to add the replicas

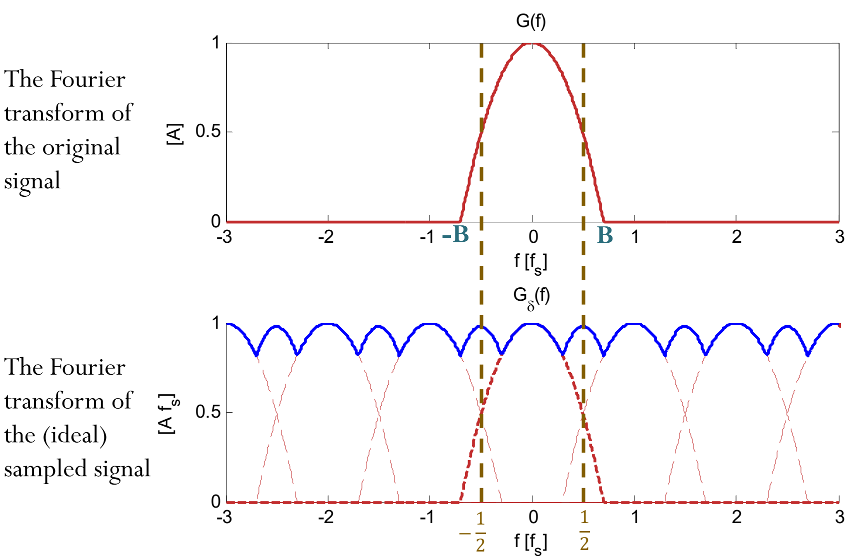


Figure 51: The Fourier transform $G_\delta(f)$ of $g_\delta(t)$ when $B > f_s/2$

6.26. Remarks:

(a) $G_\delta(f)$ is "periodic" (in the frequency domain) with "period" f_s.

- So, it is sufficient to look at $G_\delta(f)$ between $\pm\frac{f_s}{2}$

(b) The MATLAB script `plotspect` that we have been using to visualize magnitude spectrum also relies on sampled signal. Its frequency domain plot is between $\pm\frac{f_s}{2}$.

(c) Although this sampling technique is "ideal" because it involves the use of the δ-function. We can extract many useful conclusions.

(d) One can also study the discrete-time Fourier transform (DTFT) to look at the frequency representation of the sampled signal.

6.3 Reconstruction

Definition 6.27. Reconstruction (interpolation) is the process of reconstructing a continuous time signal $g(t)$ from its samples.

6.28. From (83), we see that when the sampling frequency f_s is large enough, the replicas of $G(f)$ will not overlap in the frequency domain. In such case, the original $G(f)$ is still intact and we can use a low-pass filter with gain T_s to recover $g(t)$ back from $g_\delta(t)$.

6.29. To prevent aliasing (the corruption of the original signal because its replicas overlaps in the frequency domain), we need

Theorem 6.30. A baseband signal g whose spectrum is band-limited to B Hz ($G(f) = 0$ for $|f| > B$) can be reconstructed (interpolated) exactly (without any error) from its sample taken uniformly at a rate (sampling frequency/rate) $f_s > 2B$ Hz (samples per second).[6, p 302]